ON THE REDUCTION PRINCIPLE IN CRITICAL CASES OF STABILITY OF THE MOTION OF TIME-LAG SYSTEMS

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The problem of the stability of the steady-state motion described by ordinary differential equations with time-lag is considered in the case when the characteristic equation of the first-approximation system has m roots with zero real parts and no roots with positive real parts.

It is shown that, as in the case of ordinary equations [1], this problem is equivalent to the problem of the stability of motion of a certain finite-dimensional subsystem of order m obtained by selecting the critical degrees of freedom. I n this connection the approach to the theory of critical cases in problems with retardation, worked out by Shimanov [2 and 3], is developed.

1. Formulation of the problem. Let us consider a system whose perturbed motion is described by equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + A_\tau x(t-\tau) + X(x(t), x(t-\tau)) \tag{1.1}$$

Here x is an n-vector, τ = cost > 0 is the magnitude of the lag, A, A_τ are constant matrices of the corresponding dimensions, $X(x,y)$ is an n-vector-function which in the region

$$\|x\| < H, \qquad \|y\| < H \qquad (H = \text{const}) \tag{1.2}$$

satisfies the Lipschitz condition

$$\|X(x^{(1)}, y^{(1)}) - X(x^{(2)}, y^{(2)})\| \leqslant q\,(\|x^{(1)} - x^{(2)}\| + \|y^{(1)} - y^{(2)}\|) \tag{1.3}$$

with a small quantity q

$$q = L\,(\|x^{(1)}\| + \|x^{(2)}\| + \|y^{(1)}\| + \|y^{(2)}\|)^\gamma \tag{1.4}$$

where L and γ are positive constants. Conditions (1.3) and (1.4) characterize the nonlinearity of the component $X(x(t), x(t-\tau))$ in (1.1).

In (1.2) to (1.4), as everywhere in what follows,

$$\|v\| = \Big(\sum_{i=1}^{n} v_i^2\Big)^{1/2} \tag{1.5}$$

Let us consider the space $C_{[-\tau,0]}$ of continuous functions given on the interval $[-\tau, 0]$. In [4] (p.157) it is shown that in the space $C_{[-\tau,0]}$ Equation (1.1) corresponds to the differential-operator equation

$$dx_t(\vartheta)/dt = Px_t(\vartheta) + R[x_t(0), x_t(-\tau)] \tag{1.6}$$

Here $x_t(\vartheta) = x(t+\vartheta)\,(-\tau \leqslant \vartheta \leqslant 0)$ is the segment of the trajectory corresponding to the instant t in the system (1.1); the operator P is defined by the equality

$$Px(\vartheta) = \begin{cases} dx(\vartheta)/d\vartheta & (-\tau \leqslant \vartheta < 0) \\ Ax(0) + A_\tau x(-\tau) & (\vartheta = 0) \end{cases} \tag{1.7}$$

The nonlinear operator

$$R[x(0), x(-\tau)] = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0) \\ X(x(0), x(-\tau))) & (\vartheta = 0) \end{cases} \tag{1.8}$$

Everywhere in the following it is assumed that the argument ϑ varies within the limits $-\tau \leqslant \vartheta \leqslant 0$. The spectrum of operator P in (1.7) is determined ([4](p.164)) by the roots of the characteristic equation

$$\det[A - \lambda E + A_\tau e^{-\lambda\tau}] = 0 \tag{1.9}$$

Let us assume that Equation (1.9) has m roots λ_i with $\operatorname{Re}\lambda_i = 0$ $(i = 1,\ldots, m)$, while the remaining roots have negative real parts. It is known that m is a finite number.

Let l_σ Jordan sequences of root elements of the operator P (1.7) correspond to the eigenvalue λ_σ of multiplicity m_σ

$$d_\sigma[\vartheta;\, k_\sigma\; i_\sigma]$$

$$(i_\sigma = 0, \ldots, m_\sigma[k_\sigma];\; k_\sigma = 1, \ldots, l_\sigma;\; l_\sigma + m_\sigma[1] + \ldots + m_\sigma[l_\sigma] = m_\sigma;\; m_1 + \ldots + m_r = m)$$

(the value $i_\sigma = 0$ corresponds to the free term) [5].

Then, the operator conjugate to (1.7) [5]

$$-P^* x(-\vartheta) = \begin{cases} dx(-\vartheta)/d(-\vartheta) & (-\tau \leqslant \vartheta < 0) \\ Ax(0) + A_\tau' x(\tau) & (\vartheta = 0) \end{cases} \tag{1.10}$$

has r eigenvalues $\lambda_\sigma^* = -\lambda_\sigma$ of multiplicity m_σ; to each of them there corresponds l_σ Jordan sequences of root elements of the operator P^* (1.10)

$$d_\sigma^*[-\vartheta;\, k_\sigma,\, i_\sigma]$$

$$(i_\sigma = 0,\ldots, m_\sigma[k_\sigma];\; k_\sigma = 1,\ldots, l_\sigma;\; l_\sigma + m_\sigma[1] + \ldots + m_\sigma[l_\sigma] = m_\sigma;\; m_1 + \ldots + m_r = m)$$

(In (1.10) and everywhere in the following the sign / denotes transposition.)

Let us denote [5] the quantity $(\gamma(-\vartheta) \in C_{[0,\tau]})$ by the symbol (φ, γ)

$$(\varphi, \gamma) = \varphi'(0)\gamma(0) + \int_{-\tau}^{0} \varphi'(\vartheta) A_\tau' \gamma(\vartheta + \tau)\, d\vartheta \tag{1.11}$$

and we shall represent every element $\varphi(\vartheta) \in C_{[-\tau,0]}$ in the form

$$\varphi(\vartheta) = z(\vartheta) + \sum_{\sigma=1}^{r} \sum_{k_\sigma=1}^{l_\sigma} \sum_{i_\sigma=0}^{m_\sigma[k_\sigma]} d_\sigma[\vartheta; k_\sigma, i_\sigma]\, y_\sigma[k_\sigma, i_\sigma] \tag{1.12}$$

by setting the constants

$$y_\sigma[k_\sigma, i_\sigma] = f_\sigma[\varphi(\vartheta); k_\sigma, m_\sigma[k_\sigma] - i_\sigma] = (\varphi, d_\sigma^*[k_\sigma, m_\sigma[k_\sigma] - i_\sigma]) \tag{1.13}$$

From (1.12) and (1.13) it follows that $z(\vartheta)$ is an element of the functional subspace L_f defined by the equalities

$$(L_f) \quad f_\sigma[z(\vartheta); k_\sigma, i_\sigma] = 0, \quad (i_\sigma = 0, \ldots, m_\sigma[k_\sigma]; k_\sigma = 1, \ldots, l_\sigma; \sigma = 1, \ldots, r) \tag{1.14}$$

By virtue of (1.12), for the element $x_t(\vartheta)$ of the trajectory of system (1.1) we have

$$x_t(\vartheta) = z_t(\vartheta) = \sum_{\sigma=1}^{r} \sum_{k_\sigma=1}^{l_\sigma} \sum_{i_\sigma=0}^{m_\sigma[k_\sigma]} d_\sigma[\vartheta; k_\sigma, i_\sigma] y_\sigma[t; k_\sigma, i_\sigma] \tag{1.15}$$

where $y_\sigma[t; k_\sigma, i_\sigma] = f_\sigma[x_t(\vartheta); k_\sigma, m_\sigma[k_\sigma] - i_\sigma]$. The variables $z_t(\vartheta), y(t) = \{y_i(t)$ $(i = 1, \ldots, m)\} = \{y_\sigma[t; k_\sigma, i_\sigma], i_\sigma = 0, \ldots, m_\sigma[k_\sigma];\; k_\sigma = 1, \ldots, l_\sigma (\sigma = 1, \ldots, r)\}$ satisfy Equations

$$dy(t)/dt = Gy(t) + Y[y(t), z_t(0), z_t(-\tau)] \tag{1.16}$$

$$dz_t(\vartheta)/dt = Pz_t(\vartheta) + Z[y(t), z_t(0), z_t(-\tau), \nu] \tag{1.17}$$

Here

$$G = \begin{bmatrix} \lambda_1 & \alpha_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \alpha_2 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & \lambda_m \end{bmatrix} = \text{const} \qquad (m \times m \text{ - matrix}) \tag{1.18}$$

The coefficients α_i equal either zero or unity depending on the structure of the chosen part of the spectrum $\{\lambda_i;\ \mathrm{Re}\,\lambda_i = 0;\ i = 1, \ldots, m\}$ of the operator P defined in (1.7); the m-vector-function $Y[y, z(0), z(-\tau)]$ and the real n-vector-operator $Z[y, z(0), z(-\tau), \vartheta]$ are determined thus:

$$Y[y, z(0), z(-\tau)] = \{Y_i (i = 1, \ldots, m)\} = \Big\{Y_\sigma[y, z(0), z(-\tau); k_\sigma, i_\sigma] =$$

$$= d_\sigma^*[0; k_\sigma, m_\sigma[k_\sigma] - i_\sigma]\, X\Big(z(0) + \sum_{\sigma=1}^{r} \sum_{k_\sigma=1}^{l_\sigma} \sum_{i_\sigma=0}^{m_\sigma[k_\sigma]} d_\sigma[0; k_\sigma, i_\sigma]\, y_\sigma[k_\sigma, i_\sigma] \tag{1.19}$$

$$z(-\tau) + \ldots\Big);\ i_\sigma = 0, \ldots, m_\sigma[k_\sigma];\ k_\sigma = 1, \ldots, l_\sigma (\sigma = 1, \ldots, r)\Big\}$$

$$Z[y, z(0), z(-\tau), \vartheta] = R\Big[z(0) + \sum_{\sigma=1}^{r} \sum_{k_\sigma=1}^{l_\sigma} \sum_{i_\sigma=0}^{m_\sigma[k_\sigma]} d_\sigma[0; k_\sigma, i_\sigma]\, y_\sigma[k_\sigma, i_\sigma] \tag{1.20}$$

$$z(-\tau) + \ldots\Big] - \sum_{\sigma=1}^{r} \sum_{k_\sigma=1}^{l_\sigma} \sum_{i_\sigma=0}^{m_\sigma[k_\sigma]} d_\sigma[\vartheta; k_\sigma\, i_\sigma]\, Y_\sigma[y, z(0), z(-\tau); k_\sigma, i_\sigma]$$

Let the chosen part of the spectrum contain q zero and p purely imaginary numbers

$$\{\lambda_{r_i} = 0 \ (i = 1, \ldots, q\},\quad \{\lambda_{s_k};\ \mathrm{Re}\,\lambda_{s_k} = 0; \lambda_{s_k} \neq 0 \ (k = 1, \ldots, p)\},\quad p + q = m$$

By passing to real form of the equations, we introduce the notations

$$v = \{v_i\,(i = 1, \ldots, m)\} = \{y_{r_j}\,(j = 1, \ldots, q);\ \operatorname{Re} y_{s_k},\ \operatorname{Im} y_{s_k}\,(k = 1, \ldots, p/2)\} \tag{1.21}$$

$$F[v, z(0), z(-\tau)] = \{F_i;\quad i = 1, \ldots, m\} = \tag{1.22}$$

$$= \{Y_{r_j};\ j = 1, \ldots, q;\ \operatorname{Re} Y_{s_k};\ \operatorname{Im} Y_{s_k};\ k = 1, \ldots, p/2 \quad \text{for } y_{r_j} = v_j\,(j = 1, \ldots, q)$$

$$\operatorname{Re} y_{s_k} = v_{q+k};\ \operatorname{Im} y_{s_k} = v_{q+k+1}, \qquad (k = 1, \ldots, p/2)\}$$

$$Z[v, z(0), z(-\tau), \vartheta] = \tag{1.23}$$

$$= \{Z[y, z(0), z(-\tau), \vartheta] \quad \text{for } y_{r_j} = v_j;\ Y_{r_j} = F_j \quad (j = 1, \ldots, q);\ \operatorname{Re} y_{s_k} = v_{q+k}$$

$$\operatorname{Im} y_s = v_{q+k+1},\ \operatorname{Re} Y_{s_k} = F_{q+k},\ \operatorname{Im} Y_s = F_{q+k+1}\ (k = 1, \ldots, p/2)\}$$

Then the equations of motion (1.16) and (1.17) can be represented as

$$dv/dt = Qv + F[v, z_t(0), z_t(-\tau)] \tag{1.24}$$

$$dz_t(\vartheta)/dt = Pz_t(\vartheta) + Z[v, z(0), z(-\tau), \vartheta] \tag{1.25}$$

Here the constant $m \times m$ matrix Q has the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0 & \alpha_{r_1} & 0 & \ldots & 0 \\ 0 & 0 & \alpha_{r_2} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots & \alpha_{r_q} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{1.26}$$

$$Q_2 = \begin{bmatrix} 0 & -\operatorname{Im}\lambda_{s_1} & \alpha_{s_1} & 0 & \ldots & 0 & 0 \\ \operatorname{Im}\lambda_{s_1} & 0 & 0 & \alpha_{s_1} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 0 & -\operatorname{Im}\lambda_{s_{p/2}} \\ 0 & 0 & 0 & 0 & \ldots & \operatorname{Im}\lambda_{s_{p/2}} & 0 \end{bmatrix}$$

The problem of the stability of system (1.1) and of the system (1.24), (1.25) are equivalent.

As in the case of systems described by ordinary differential equations [1], the question arises whether the properties of stability of the "truncated" system

$$dv/dt = Qv + F[v, 0, 0] \tag{1.27}$$

determine the analogous properties of the complete system (1.24), (1.25) (i.e. of system (1.1)).

In Section 2 below we answer this question under the assumption that the right-hand sides of system (1.24), (1.25) are subject to certain restrictions. As in the case of ordinary differential equations [1] we can show that for analytic right-hand sides of system (1.1) there exists a transformation which takes the system (1.24), (1.25) to the form where these conditions are satisfied. The proof of the latter assertion is omitted here.

The proof of the reduction principle is carried out here analogously to the proof (*) of this principle in the case of ordinary equations [1].

*) Erugin has remarked [6] that the proof in [1] contains a serious inaccuracy. In the present paper this remark is taken into account and the proof has been supplemented.

2. The reduction principle. In the space $C_{[-\tau,0]}$ let us introduce the metric

$$\|\varphi(\vartheta)\|_\tau = \sup\left(\sum_{i=1}^{n} |\varphi_i(\vartheta)|^2\right)^{1/2} \qquad (2.1)$$

By virtue of (1.3) and (1.4) the functions $F[v, z(0), z(-\tau)]$, $Z[v, z(0), z(-\tau), \vartheta]$ in the region

$$(G) \qquad \|v\| < H_1, \qquad \|z(\vartheta)\|_\tau < H_1 \qquad (2.2)$$

satisfy Lipschitz conditions with small multipliers

$$\|F[v', z'(0), z'(-\tau)] - F[v'', z''(0), z''(-\tau)]\| \leqslant \qquad (2.3)$$
$$\leqslant q_1(\|v' - v''\| + \|z'(\vartheta) - z''(\vartheta)\|_\tau)$$
$$q_1 = L_1(\|v'\| + \|v''\| + \|z'(\vartheta)\|_\tau + \|z''(\vartheta)\|_\tau)^\gamma \quad (L_1 = \text{const} > 0)$$
$$\|Z[v', z'(0), z'(-\tau), \vartheta] - Z[v'', z''(0), z''(-\tau), \vartheta]\|_\tau \leqslant$$
$$\leqslant q_2(\|v' - v''\| + \|z'(\vartheta) - z''(\vartheta)\|_\tau) \qquad (2.4)$$
$$q_2 = L_2(\|v'\| + \|v''\| + \|z'(\vartheta)\|_\tau + \|z''(\vartheta)\|_\tau)^\gamma \quad (L_2 = \text{const} > 0)$$

Here the quantity γ is the same as in (1.4), and the positive constant H_1 is computed in a specific manner from the constant H of (1.2) in accordance with (1.12), (1.5) and (2.1).

We cite some definitions [1]. Consider the system of equations

$$dv / dt = F[v, t] \qquad (2.5)$$

where the continuous m-vector-function $F[v, t]$ satisfies in region (2.1) the Lipschitz condition

$$\|F[v^{(1)}, t] - F[v^{(2)}, t]\| \leqslant L\|v^{(1)} - v^{(2)}\| \qquad (L = \text{const} > 0)$$

and $F[0, t] \equiv 0$. In region (2.1) let $F[v, t]$ admit of the representation

$$F[v, t] = F_1[v, t] + F_2[v, t] \qquad (2.6)$$

where $F_2[v,t]$ denotes the set of terms of order in v higher than N and, moreover,

$$\|F_2[v, t]\| \leqslant K\|v\|^{N+\gamma} \qquad (K = \text{const} > 0, N = \text{const} \geqslant 1) \qquad (2.7)$$

where the quantity γ is the same as that in conditions (2.3) and (2.4).

D e f i n i t i o n 2.1. The unperturbed motion $v = 0$ of system (2.5) is said to be stable regardless of terms of order higher than N, if for any positive number ε, as small as desired, there exists a positive number $\eta(\varepsilon, K)$, depending only on ε and K, such that for all solutions of Equation (2.5) for which at the initial instant $t = 0$ the conditions

$$\|v(0)\| \leqslant \eta(\varepsilon, K) \qquad (2.8)$$

are satisfied, for all $t > 0$ the inequality

$$\|v(t)\| < \varepsilon \qquad (2.9)$$

will be satisfied for any choice of the function $F_2[v, t]$ satisfying estimate (2.7) in region (2.1).

D e f i n i t i o n 2.2. The unperturbed motion $v = 0$ of system (2.5) is said to be unstable regardless of terms of order higher than N, if for the same conditions on the function $F_2[v, t]$ there exists a positive number $\varepsilon(K)$, depending only on K, such that no matter how small the number $\eta > 0$ is, there is found a vector $v^0(K, \eta)$, depending only on K and η, for which $\|v^0(K, \eta)\| \leqslant \eta$, and, the solution $v(t)$ of Equations (2.5) with initial condition

$$v\,(0) = v^{\circ}\,(K, \eta) \tag{2.10}$$

will satisfy the equality

$$\|\, v\,(t_1)\,\| = \varepsilon\,(K) \tag{2.11}$$

at some instant $t = t_1 = \text{const}$.

T h e o r e m 2.1. Let the right-hand sides of system (1.24), (1.25) satisfy in the region (2.1), besides the conditions (2.3) and (2.4), also the conditions:

1) The unperturbed motion $v = 0$ of the "truncated" system (1.27) is stable (asymptotically stable) or unstable regardless of terms of order higher than N in the sense of Definitions 2.1 and 2.2.

2) The operator $Z\,[v, 0, 0, \vartheta]$ satisfies the estimate

$$\|Z\,[v, 0, 0, \vartheta]_\tau \leqslant L_4 \|\, v\,\|^{N+\gamma_1} \qquad (L_4 = \text{const} > 0,\ \gamma_1 = \text{const} > 0) \tag{2.12}$$

Then the property of stability of system (1.24), (1.25) (and, consequently, of system (1.1)) coincides with the property of stability of the "truncated" system (1.27).

P r o o f . Let the unperturbed motion $v = 0$ of the "truncated" system (1.27) be stable (in the sense of Definition 2.1).

Without loss of generality we can take it that the matrix (1.26)

$$Q = \{Q\}_{\alpha_{i=0}} + Q^* \tag{2.13}$$

where the matrix Q^* satisfies the estimate

$$\|\, Q^* \,\| < \varepsilon_1 \qquad (\varepsilon_1 \text{ arbitrary positive number}) \tag{2.14}$$

Indeed, this always can be achieved by a nonsingular linear real transformation of the vector variable v .

In the subspace L_f (1.14) let us consider Equations

$$\frac{du_t\,(\vartheta)}{dt} = Pu_t\,(\vartheta), \qquad u_t\,(\vartheta) \in L_f\,(1.14) \tag{2.15}$$

Any solution $u_t(\vartheta)$ of system (2.15) (with (c $u_0(\vartheta) \in L_f\,(1.14)$) satisfies the estimate [5]

$$\|\, u_t\,(\vartheta)\|_\tau \leqslant B\, \|\, u_0\,(\vartheta)\|_\tau \exp\,[-\beta t] \qquad (B,\ \beta = \text{const} > 0) \tag{2.16}$$

But then on the basis of the results in [4] (pp.191-193) in the subspace L_f (1.14) a continuous functional $V\,[u\,(\vartheta)]$, satisfying the estimates

$$c_1 \|\, u\,(\vartheta)\|_\tau^2 \leqslant V\,[u\,(\vartheta)] \leqslant c_2 \|\, u\,(\vartheta)\|_\tau^2 \tag{2.17}$$

$$\limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.15)} \leqslant -\, c_3 \|\, u\,(\vartheta)\|_\tau^2 \tag{2.18}$$

$$|\, V\,[u^{(1)}\,(\vartheta)] - V\,[u^{(2)}\,(\vartheta)]| \leqslant c_4 \|\, u^{(1)}\,(\vartheta) - u^{(2)}\,(\vartheta)\|_\tau \max\,(\|\, u^{(1)}\,(\vartheta)\|_\tau, \|\, u^{(2)}\,(\vartheta)\|_\tau) \tag{2.19}$$

can be constructed.

Here $c_1, \ldots, c_4$ are positive constants.

Let us make a change of variables in (1.24), (1.25) by setting

$$z\,(\vartheta) = \|\, v\,\|^N \cdot \xi\,(\vartheta) \tag{2.20}$$

Transformation (2.20) will be used only in the region $G_\xi \subset G\,(2.1)$ of the space $\{v, z\,(\vartheta)\}$, where the quantity $\|\,\xi\,(\vartheta)\|_\tau$ is bounded

$$(G_\xi) \quad \|\,\xi\,(\vartheta)\| < H_2 \tag{2.21}$$

Here μ_2 is some positive constant.

In the variables $v(t)$, $\xi_t(\vartheta)$ Equations (1.24), (1.25) take the forms

$$\frac{dv}{dt} = Qv + F[v, 0, 0] + F^*[v, \xi_t(0), \xi_t(-\tau)] \tag{2.22}$$

$$\frac{d\xi_t(\vartheta)}{dt} = P\xi_t(\vartheta) - N\|v\|^{-2} v'Qv\xi_t(\vartheta) + Z^*[v, \xi_t(0), \xi(-\tau), \vartheta] \tag{2.23}$$

where the functions

$$F^*[v, \xi(0), \xi(-\tau)] = F[v, \|v\|^N \xi(0), \|v\|^N \xi(-\tau)] - F[v, 0, 0] \tag{2.24}$$

$$Z^*[v, \xi(0), \xi(-\tau), \vartheta] = \|v\|^{-N} Z[v, \|v\|^N \xi(0), \|v\|^N \xi(-\tau), \vartheta] - \\ - N\|v\|^{-2} v'F[v, \|v\|^N \xi(0), \|v\|^N \xi(-\tau)] \tag{2.25}$$

satisfy the estimates

$$\|F^*[v, \xi(0), \xi(-\tau)]\| \leqslant L_5 \|v\|^{N+\gamma} \qquad (L_5 = \text{const} > 0) \tag{2.26}$$

$$Z^*[0, \xi(0), \xi(-\tau), \vartheta] \equiv 0 \tag{2.27}$$

in the region G_ξ (2.21), by virtue of conditions (2.3), (2.4) and (2.12).

Let there be given an arbitrary positive number $\varepsilon < \mu_2$. In the region G_ξ (2.21) let us consider the hypersurface M_1

$$(M_1) \qquad\qquad V[\xi(\vartheta)] = l_1(\varepsilon) \tag{2.28}$$

where $l_1(\varepsilon)$ is a constant satisfying the condition

$$0 < l_1(\varepsilon) < c_1\varepsilon^2 \tag{2.29}$$

Then, by virtue of (2.17) the estimates

$$\mu(\varepsilon) \leqslant \|\xi(\vartheta)\|_\tau < \varepsilon \quad \text{for } \xi(\vartheta) \text{ from } M_1 \tag{2.30}$$

are satisfied, where $\mu(\varepsilon)$ is some positive constant.

Let us compute lim sup $\Delta V/\Delta t$ as $\Delta t \to +0$ by virtue of (2.23) on the hypersurface M_1 (2.28), taking estimates (2.17) to (2.19) and (2.30) into account. For brevity we introduce the notations: $\xi_t[\vartheta; (2.23)]$ is an element of the trajectory of system (2.23) and $\xi_t[\vartheta; (2.15)]$ is an element of the trajectory of system (2.15).

We have

$$\limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.23)} \leqslant \limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.15)} + \limsup_{\Delta t \to +0} \frac{1}{\Delta t} \,|V[\xi_{t+\Delta t}[\vartheta; (2.23)]] -$$

$$- V[\xi_{t+\Delta t}[\vartheta; (2.15)]]| \leqslant -c_3\|\xi_t(\vartheta)\|_\tau^2 + c_4 \limsup_{\Delta t \to +0} \frac{1}{\Delta t} [\|\xi_{t+\Delta t}[\vartheta; (2.23)] -$$

$$- \xi_{t+\Delta t}[\vartheta; (2.15)]\|_\tau \max(\|\xi_{t+\Delta t}[\vartheta; (2.23)]\|_\tau, \|\xi_{t+\Delta t}[\vartheta; (2.15)]\|_\tau)] \leqslant -c_3\|\xi_t(\vartheta)\|_\tau^2 +$$

$$+ N\|v\|^{-2}|v'Qv| \,\|\xi_t(\vartheta)\|_\tau^2 + c_4\|\xi_t(\vartheta)\|_\tau \|Z^*[v, \xi_t(0), \xi_t(-\tau), \vartheta]\|_\tau$$

Here it was assumed that $\xi_t[\vartheta; (2.23)] = \xi_t[\vartheta; (2.15)]$.

Taking into account that $v'\{Q\}v \equiv 0$, when $d_i = 0$ and choosing the number ε_1 in (2.14) from the condition $\varepsilon_1 < c_3/N$ on the hypersurface M_1 (2.28) we obtain the estimate

$$\left\{\limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.23)}\right\}_{\xi(\vartheta) \in M_1} \leqslant -(c_3 - N\varepsilon_1)\mu^2(\varepsilon) + c_4\varepsilon\|Z^*[v, \xi(0), \xi(-\tau), \vartheta]\|_\tau \tag{2.31}$$

From the property (2.27) of the function $Z^*[v, \xi(0), \xi(-\tau), \vartheta]$ and from estimate (2.31) it follows that there exists a positive number $h(\varepsilon) < \varepsilon$ such that

$$\left\{\limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.23)}\right\}_{\xi(\vartheta) \in M_1} < 0 \qquad \text{for } \|v\| \leqslant h(\varepsilon) < \varepsilon \tag{2.32}$$

Let $\xi_t^\circ(\vartheta)$ be an arbitrary function satisfying the condition

$$\xi_t^\circ(\vartheta) \in G_\xi\,(2.21) \quad \text{for} \quad t \geqslant 0 \tag{2.33}$$

In (2.22) let us put $\xi_t(\vartheta) = \xi_t^\circ(\vartheta)$. We get quations

$$dv\,/\,dt = Qv + F_1[v] + F^*[v, t] \tag{2.34}$$

Here the function $F_1[v]$ denotes the set of terms of order higher than N in the function $F[v,0,0]$ of (1.27) and, consequently, by virtue of (2.26) and of condition (1) of the theorem, the function

$$F^*[v, t] = F[v, 0, 0] - F_1[v] + F^*[v, \xi_t^\circ(0), \xi_t^\circ(-\tau)]$$

satisfies the estimate

$$\|F^*[v, t]\| \leqslant K_1 \|v\|^{N+\gamma} \tag{2.35}$$

when $\xi_t^\circ \in G_\xi$ (2.21)

Here K_1 is a positive constant depending only on the structure of Equation (2.22) and not depending on any particular choice of $\xi_t^\circ(\vartheta) \in G_\xi$ (2.21) In accordance with the condition for the stability of the "truncated" system (1.27) regrdless of terms of order higher than N, from the number $h(\varepsilon)$ we can find a positive number $\delta(h(\varepsilon), K_1)$ such that for all solutions of (2.34) which satisfy

$$\|v(0)\| \leqslant \delta(h(\varepsilon), K_1) \tag{2.36}$$

at $t = 0$, the estimate

$$\|v(t)\| < h(\varepsilon) \tag{2.37}$$

will hold for all $t > 0$.

However, if the function $\xi_t^\circ(\vartheta)$ satisfies (2.33) on the interval $[0, T = \mathrm{const}]$, then all the solutions of (2.34), which at the initial instant $t = 0$ satisfy inequality (2.36), will satisfy inequality (2.37) on this same interval.

In the space $\{v, z(\vartheta)\}$ let us consider the hypersurface M_2

$$(M_2) \qquad V[z(\vartheta)] = l_2(\varepsilon) \tag{2.38}$$

where $l_2(\varepsilon)$ is some sufficiently small positive number which will be used later. In the space $\{v, z(\vartheta)\}$ let us define two regions G_1 and G_2 sy setting

$$(G_1) \qquad \|v\| \leqslant \eta(\varepsilon), \quad V[\xi(\vartheta)] \leqslant l_1(\varepsilon) \tag{2.39}$$

$$(G_2) \quad \|v\| \leqslant \eta(\varepsilon), \quad V[\xi(\vartheta)] \geqslant l_1(\varepsilon), \quad V[z(\vartheta)] \leqslant l_2(\varepsilon) \tag{2.40}$$

Here $\eta(\varepsilon)$ is a positive number satisfying conditions

$$\eta(\varepsilon) \leqslant \delta(h(\varepsilon), K_1) \tag{2.41}$$

$$V[\xi(\vartheta)] < l_1(\varepsilon) \quad \text{for} \quad \|\xi(\vartheta)\|_\tau \leqslant \eta(\varepsilon) \tag{2.42}$$

Such a choice of the $\eta(\varepsilon)$ is possible by virtue of property (2.17).

From (2.17), (2.39) and (2.40) it follows that the union G^* of regions G_1 and G_2 $(G^* = G_1 \cup G_2)$ contains the null element $\{v = 0, z(\vartheta) = 0\}$ as an interior point. For sufficiently small $l_2(\varepsilon)$ we have $G^* \subset G$ (2.1).

Let us first assume that the initial perturbations belong to the region G_1 (2.39). Since $G_1 \subset G_\xi$ (2.21) the equations of motion can be considered in the forms (2.22) and (2.23). Let $v(t)$ and $\xi_t(\vartheta)$ be arbitrary solutions of (2.22), (2.23), for which the inequalities

$$\|v(0)\| \leqslant \eta(\varepsilon), \qquad \|\xi_0(\vartheta)\|_\tau \leqslant \eta(\varepsilon) \tag{2.43}$$

are satisfied at the initial instant $t = 0$.

From (2.43) it follows that for all $t > 0$ the inequalities

$$\|v(t)\| < \varepsilon, \qquad \|\xi_t(\vartheta)\|_\tau < \varepsilon \tag{2.44}$$

are satisfied.

Indeed, the second inequality in (2.44), satisfied for $t = 0$, will by continuity be satisfied for sufficiently small t. Let $t = T$ be the first instant at which $\|\xi_t(\vartheta)\|_\tau = \varepsilon$. But then, on the basis of (2.28) to 2.30),

$$V[\xi_T(\vartheta)] > l_1(\varepsilon) \tag{2.45}$$

whence by virtue of the choice of $\eta(\varepsilon)$ it follows from (2.42) that there exists an instant $t = t_1 \in (0, T)$ such that the conditions

$$\xi_{t_1}(\vartheta) \in M_1, \qquad \left\{\limsup_{\Delta t \to +0}\left(\frac{\Delta V}{\Delta t}\right)\right\}_{\xi_{t_1} \in M_1} \geqslant 0 \tag{2.46}$$

will be satisfied simultaneously.

The function $\xi_t(\vartheta)$ satisfies condition (2.33) when $t \in (0, T)$. Further, since the function $v(t)$ will be one of the solutions of Equation (2.34) for $\xi_t{}^\circ(\vartheta) = \xi_t(\vartheta)$ and since the number $\eta(\varepsilon)$ is chosen in accordance with (2.41), inequality (2.37) will be satisfied for $v(t)$ in the whole time interval $(0, T)$ and, consequently, also the inequality (2.32). The latter contradicts (2.46). Thus, inequality (2.45) is satisfied for all $t > 0$. But then, from the preceding discussion it follows that the function $v(t)$ satisfies inequality (2.37) for all $t > 0$, whence by virtue of the condition $h(\varepsilon) < \varepsilon$ it follows that the first estimate in (2.44) holds for all $t > 0$.

Thus, we have proved the conditional stability of the unperturbed motion $v(t) = 0$, $z_t(\vartheta) = 0$ of the system (1.24), (1.25) with respect to the initial perturbations $z_0(\vartheta)$ from the region G_1 (2.39) which are constrained by the conditions (2.43).

Let us show that from such a conditional stability follows the stability of the motion $v(t) = 0$, $z_t(\vartheta) = 0$ with respect to any sufficiently small initial perturbations.

In (2.40) we choose the number $l_2(\varepsilon)$ from the conditions

$$l_2(\varepsilon) < \varepsilon, \qquad l_2(\varepsilon) < c_1 c_2 \mu^2(\varepsilon)\ \eta^{2N}(\varepsilon) \tag{2.47}$$

$$\limsup_{\Delta t \to +0}\left(\frac{\Delta V[z_t(\vartheta)]}{\Delta t}\right)_{(1.24),(1.25)} < 0 \tag{2.48}$$

for $z(\vartheta)$ from M_2 (2.38) in region G_2 (2.40).

The second condition in (2.47) signifies that in the space $\{v, z(\vartheta)\}$ the hypersurface $V[z(\vartheta)] = l_2(\varepsilon)$ (2.38) intersects the hypersurface $V[\xi(\vartheta)] = l_1(\varepsilon)$ (2.28) when $\|v\| \leqslant \eta(\varepsilon)$, so that from the region G_2 (2.40) it is impossible to derive the element $\{v, z(\vartheta)\}$, by continuously deforming it and yet not intersecting hypersurface (2.28) or (2.38).

It is possible to select the number $l_2(\varepsilon)$ so as to satisfy condition (2.48) also. Indeed, let us compute $\limsup \Delta V[z_t(\vartheta)]/\Delta t$ as $\Delta t \to +0$ along the motion of system (1.24), (1.25). Taking estimates (2.17) to (2.19) into account here, we get

$$\limsup_{\Delta t \to +0}\left(\frac{\Delta V[z_t(\vartheta)]}{\Delta t}\right)_{(1.24),(1.25)} \leqslant -c_3\|z_t(\vartheta)\|_\tau^2 +$$

$$+ c_4\|z_t(\vartheta)\|(\|Z[v, 0, 0, \vartheta]\|_\tau + \|Z[v, z_t(0), z_t(-\tau), \vartheta] - Z[v, 0, 0, \vartheta]\|_\tau) \tag{2.49}$$

Since the condition

$$\|v\|^N \leqslant \frac{1}{\sqrt{c_1}\mu_*(\varepsilon)}\|z(\vartheta)\|_\tau \tag{2.50}$$

is satisfied in the region G_2 (2.40) by virtue of (2.17), then from the estimates (2.4), (2.12), (2.38), (2.17), (2.49) follows the possibility of choosing an $l_2(\varepsilon)$ satisfying (2.47) and (2.48).

Let us consider an arbitrary motion $v(t)$, $z_t(\vartheta)$ of system (1.24), (1.25) whose initial perturbation $\{v(0), z_0(\vartheta)\}$ lies in the region G_2 (2.40).

By virtue of the choice of $l_2(\varepsilon)$ in accordance with (2.47) and (2.48), the motion $v(t)$ $z_t(\vartheta)$, starting from the point $\{v(0), z_0(\vartheta)\} \in G_2$ either remains in the region G_2 for all time, and then by the way in which the latter was constructed, the inequalities

$$\|v(t)\| < \varepsilon, \qquad \|z_t(\vartheta)\|_\tau < \varepsilon \tag{2.51}$$

will be satisfied for all $t > 0$ (moreover, $\lim v(t) = 0$, $\lim z_t(\vartheta) = 0$ as $t \to \infty$), or the motion $v(t)$, $z_t(\vartheta)$ leaves the region G_2 (2.40) through the hypersurface (2.28) and here necessarily when $\|v\| \leqslant \eta(\varepsilon)$. In the latter case, by what was proved earlier, when the motion $v(t)$, $z_t(\vartheta)$ falls into the region G_1 (2.39) it remains for all time in the region

$$\|v\| < h(\varepsilon), \qquad V[\xi(\vartheta)] \leqslant l_1(\varepsilon) \tag{2.52}$$

Thus, the stability of the unperturbed motion $v(t) = 0$, $z_t(\vartheta) = 0$ of the system (1.24),(1.25) (and, consequently, the stability of the unperturbed motion $x = 0$ of system (1.1)) follows from the stability (in the sense of definition 2.1) of the motion $v = 0$ of the "truncated" system (1.27).

The geometric meaning of the reasoning carried out is clarified in Fig.1.

Now let the unperturbed motion of system (1.27) be asymptotically stable.

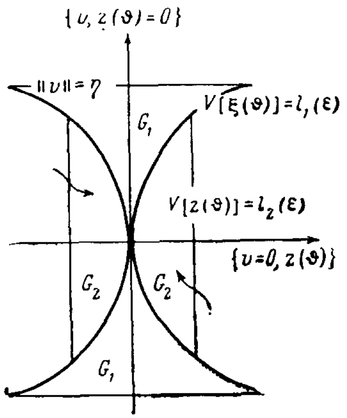

Fig. 1

Let us consider an arbitray motion $v(t)$, $\xi_t(\vartheta)$ of system (2.22),(2.23) with the initial conditions

$$\|v(0)\| \leqslant \eta, \qquad \|\xi_0(\vartheta)\|_\tau \leqslant \eta \tag{2.53}$$

where η is a sufficiently small positive number. By virtue of the conditions of the theorem and on the basis of what has been proved, the unperturbed motion $v = 0$, $\xi_t(\vartheta) = 0$ of system (2.22),(2.23) is stable and, consequently, the function $\xi^t(\vartheta)$ satisfies condition (2.33). But when $\xi_t^\circ(\vartheta) = \xi_t(\vartheta)$, the unperturbed motion $v = 0$ of system (2.34) will be asymptotically stable. And since the function $v(t)$ is one of the solutions of (2.34) when $\xi_t^\circ(\vartheta) = \xi_t(\vartheta)$, then it necessarily satisfies the limit relation

$$\lim v(t) = 0 \quad \text{for } t \to \infty \tag{2.54}$$

for sufficiently small η in (2.53).

We can show that $\xi_t(\vartheta)$ also satisfies (2.54).

Let l be an arbitrary positive number as small as desired. Consider the set M_3 of functions $\xi(\vartheta)$

$$(M_3) \qquad \xi(\vartheta) \in G_\xi\ (2.21), \qquad V[\xi(\vartheta)] \geqslant l \tag{2.55}$$

Let us substitute the function $v(t)$ into (2.23) and, under condition (2.54) compute $\limsup \Delta V/\Delta t$ as $\Delta t \to +0$ by virtue of the equations obtained, one of whose solutions will be function $\xi_t(\vartheta)$. By repeating the arguments used to derive the estimate (2.32) and by taking the limit relation (2.54) for function $v(t)$ into account, we conclude that there exists such an instant $t = t_1$ beginning with which the inequality

$$\left\{\limsup_{\Delta t \to +0} \left(\frac{\Delta V}{\Delta t}\right)_{(2.23)}\right\}_{\xi(\vartheta) \in M_3} < -\gamma \quad \text{for } t > t_1 \tag{2.56}$$

will be satisfied for all solutions of Equation (2.23) when $v = v(t)$ and, in particular, for $\xi_t(\vartheta)$. In (2.56) the quantity γ is some positive constant. From (2.56) it follows that there exists an instant $t = t_2 > t_1$ beginning with which the inequality

$$V[\xi_t(\vartheta)] < l \tag{2.57}$$

is satisfied.

Indeed, if such an instant $t = t_2$ were not to exist, i.e. $\xi_t(\vartheta) \in M_3$ for $t > t_1$, then according to (2.56) the inequality $w(t) \leqslant w(t_1) - \gamma(t - t_1)$, would be satisfied for $w(t) = V[\xi_t(\vartheta)]$ which contradicts the estimate (2.17) when $t > w(t_1)/\gamma + t_1$.

Thus, inequality (2.57) is satisfied for all $t \geqslant t_2 = \text{const}$. But from (2.17) and (2.57), because l is arbitrary, there follows the limit relation

$$\lim \xi_t(\vartheta) = 0 \qquad \text{for } t \to \infty$$

which together with (2.54) completes the proof of the asymptotic stability of the motion $v = 0,\ z_t(\vartheta) = 0$ of system (1.24),(1.25) with respect to initial perturbations constrained by the condition $\|\xi_0(\vartheta)\|_\tau \leqslant \eta$.

From the proven conditional asymptotic stability follows the asymptotic stability of the unperturbed motion $v = 0,\ z_t(\vartheta) = 0$ of system (2.22),(2.23) with respect to any sufficiently small perturbation $v(0),\ z_0(\vartheta)$.

The proof of this assertion completely repeats the corresponding reasoning of the proof of the stability of the motion $v = 0,\ z_t(\vartheta) = 0$ (given above) and, therefore, is not carried out here.

Finally, let the unperturbed motion $v = 0$ of the "truncated" system (1.27) be unstable (in the sense of Definition 2.2).

Consider Equation (2.34). For the "truncated" system (1.27), according to the condition for instability regardless of terms of order higher than N there exists a positive number $\varepsilon(K_1)$, depending only on K_1, such that however small the number $\eta > 0$ may be, we can find a vector $v^\circ(K_1, \eta)$, depending only on K_1 and η, for which $\|v^\circ(K_1, \eta)\| \leqslant \eta$, and in connection with this the solution $v(t)$ of Equation (2.34) with the initial condition

$$v(0) = v^\circ(K_1, \eta) \tag{2.58}$$

will satisfy the equality

$$\|v(t_1)\| = h(\varepsilon) < \varepsilon \tag{2.59}$$

at some instant $t = t_1 = \text{const}$, where $h(\varepsilon)$ is defined in (2.32).

Let us suppose that the unperturbed motion $v(t) = 0,\quad z_t(\vartheta) = 0$ of system (1.24),(1.25) is stable: there exists a positive number η such that for all solutions of Equations (1.24) and (1.25) with the initial conditions

$$\|v(0)\| \leqslant \eta, \qquad \|z_0(\vartheta)\|_\tau \leqslant \eta \tag{2.60}$$

the inequalities

$$\|v(t)\| < h(\varepsilon), \qquad \|z_t(\vartheta)\|_\tau < h(\varepsilon) \tag{2.61}$$

will be satisfied for all $t > 0$.

Let us pick out the solution $v^*(t),\ z_t^*(\vartheta)$ with the initial conditions

$$v^*(0) = v^\circ(K_1, \eta) \tag{2.62}$$

$$z_0^*(\vartheta) = \varphi(\vartheta),\ (\|\varphi(\vartheta)\|_\tau < \min(\eta, \|v^\circ\|^N \sqrt{l_1(\varepsilon)} / \sqrt{c_2})) \tag{2.63}$$

From (2.63) the estimate

$$V[\xi_0^*(\vartheta)] < l_1(\varepsilon) \tag{2.64}$$

follows by virtue of (2.17) for the function $\xi_t^*(\vartheta) = \|v^*(t)\|^{-N} z_t^*(\vartheta)$ at the initial instant $t = 0$.

Then from (2.32) and (2.64) follows the inequality

$$V[\xi_t^*(\vartheta)] < l_1(\varepsilon) \quad \text{for} \quad t > 0 \tag{2.65}$$

From the latter, by virtue of (2.29), follows the estimate

$$\|\xi_t^*(\vartheta)\|_\tau < \varepsilon \quad \text{for} \quad t > 0 \tag{2.66}$$

and, consequently, $\xi_t^*(\vartheta)$ satisfies condition (2.33).

Let us set $\xi_t^\circ(\vartheta) = \xi_t^*(\vartheta)$ in (2.34). Then, the solution of the equation obtained, with the initial condition $v(0) = v^\circ(K_1, \eta)$ will coincide with the function $v^*(t)$ at least until $\|v(t)\| < h(\varepsilon)$. But condition (2.59) is satisfied for $v(t)$. The latter contradicts (2.61).

Thus, from the instability (in the sense of Definition 2.2) of the unperturbed motion $v = 0$ of the "truncated" system (1.27) there follows the instability of the unperturbed motion $v = 0$, $z_t(\vartheta) = 0$ of the system (1.24), (1.25). The theorem is proved.

N o t e 2.1. In the proof of the reduction principle we did not make use of the fact that the equations of motion (1.24) and (1.25) are stationary.

The reduction principle as given is valid also for a nonstationary system (1.24),(1.25) and, in the main, the proof repeats the one presented above if the following conditions are fulfilled:

1) The matrix $Q(t)$ is representable as a sum of a skew-symmetric matrix and a matrix with arbitrarily small coefficients.

2) The solution of Equation (2.15) satisfies the estimate

$$\|u_t(\vartheta)\|_\tau \leqslant B\|u_0(\vartheta)\|_\tau \exp[-\alpha(t - t_0)]$$

where B and α are positive constants. (This condition is equivalent to the condition of existence of a continuous functional $V[u(\vartheta), t]$, satisfying (2.17) to (2.19).

3) The operator $Z[v, 0, 0, \vartheta, t]$ satisfies estimate (2.12).

N o t e 2.2. While writing this paper the author learned that the reduction principle for time-lag systems was independently proved by Shimanov by another method.

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